Delta Matroids Whose Fundamental Graphs Are Bipartite

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ABSTRACT

We consider a generalization of finite matroids called delta matroids. This structure has been introduced simultaneously by A. Bouchet and, with pseudomatroids, by R. Chandrasekaran and S. N. Kabadi. It turns out that generalized matroids of E. Tardos and the metroids of A. Dress and T. Havel are just special cases of delta matroids. Let $\delta = (V, \mathcal{F})$ be a delta matroid. For any $F \in \mathcal{F}$, the fundamental graph G_F^δ is the simple graph on the vertex set V whose edge set is $\{xy : x \neq y,\ F \triangle \{x, y\} \in \mathcal{F}\}$. Given an *even* delta matroid δ and an element F of $\mathcal{F}$, A. Bouchet proved that G_F^δ is bipartite if and only if δ is equivalent to a matroid. We extend this result to general delta matroids in two natural ways, and we characterize the corresponding delta matroids by excluded minors.

1. INTRODUCTION

(a)

We recall the basic notions of [1]. Let $\triangle$ be the symmetric difference operator. A *delta matroid* on the finite set V is a set system $\delta = (V, \mathcal{F})$ such that the *base set* $\mathcal{F}$, $\mathcal{F} \neq \emptyset$, satisfies the *symmetric exchange axiom*,

(SEA) For $F_1, F_2 \in \mathcal{F}$ and $x \in F_1 \triangle F_2$, there exists $y \in F_2 \triangle F_1$ such that $F_1 \triangle \{x, y\} \in \mathcal{F}$.

For example, a matroid $M = (V, B)$, where B is the base set of M, satisfies the following *exchange axiom*:

(EA) For $B_1, B_2 \in B$ and $x \in B_1 - B_2$, there exists $y \in B_2 - B_1$ such that $B_1 - x + y = B_1 \triangle \{x, y\} \in B$.

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Since (EA) is a particular case of (SEA), we shall consider M as a delta matroid.

In [1], [2], and [3], A Bouchet introduced delta matroids and symmetric matroids with applications to eulerian tours of 4-regular graphs, matching theory, nonsingular principal minors of quasisymmetric matrices, and the greedy algorithm. Independently, in [5], R. Chandrasekaran and S. N. Kabadi also generalized the greedy algorithm. The g-matroids of E. Tardos [10] are also a particular case of delta matroids; they have been used to extend Edmond's intersection theorem, and so to give a matroidal proof of a supermodular coloring theorem of A. Schrijver [9]. Symmetric matroid are also used in connection with *electrically self-dual* (ESD) *matroids* of A. Recski [8] for applications in electrical network analysis.

For $A \subseteq V$, $\mathscr{F} \vartriangle A = \{F \vartriangle A : F \in \mathscr{F}\}$ is the base set of a delta matroid on V, denoted by $\delta \vartriangle A$. This new delta matroid is said to be *equivalent to* δ. Then $\delta^* = \Delta \vartriangle V$ is called the *dual* of δ. A delta matroid is said to be *even* if for any $F_1, F_2 \in \mathscr{F}$, $F_1 \vartriangle F_2$ has even cardinality. This property is preserved by equivalence. Any matroid is even.

(b)

Consider $F \in \mathscr{F}$ and the simple graph G_F^δ on the vertex set V whose edge set is $\{xy : x \neq y$ and $F \vartriangle \{x, y\} \in \mathscr{F}\}$. G_F^δ is called the *fundamental graph* defined by δ and F. For a matroid, $G = G_F^\delta$ is the bipartite graph whose color classes are F and $V - F$ and such that for $x \in V - F$, $\{y \in F : xy$ is an edge of $G\}$ is the fundamental circuit with respect to F defined by x. Conversely, the following result is proved in [2].

PROPOSITION 1.1. *Let $\delta = (V, \mathscr{F})$ be an even delta matroid and $F \in \mathscr{F}$. Then G_F^δ is bipartite if and only if δ is equivalent to a matroid.*

The purpose of this paper is to extend this result to general delta matroids in two natural ways and to characterize the corresponding delta matroids by excluded minors. For this, we consider the properties:

(BP) For any $F \in \mathscr{F}$, G_F^δ is bipartite.
(BP′) For any $F \in \mathscr{F}$, G_F^δ is bipartite *and F is a color class.*

Obviously (BP′) $\Rightarrow$ (BP), and it is easy to verify that if $X \subseteq V$ then $G_F^\delta = G_{F \vartriangle X}^{\delta \vartriangle X}$. In particular $G_F^\delta = G_{V-F}^{\delta^*}$. Thus

PROPOSITION 1.2. *The family of delta matroids satisfying* (BP) *(respectively* BP′)) *is closed by equivalence (respectively by duality).*

Consider $\mathscr{H} = \max \mathscr{F} = \max\{X \subseteq V : X \in \mathscr{F}\}$ and $\mathscr{B} = \min \mathscr{F} = \min\{X \subseteq V : X \in \mathscr{F}\}$. By (SEA), it is easy to see that $\mathscr{H}$ and $\mathscr{B}$ are collections of bases of matroids, respectively the *upper matroid*, denoted by $\mathfrak{U}$, and the *lower matroid*, denoted by $\mathfrak{L}$. The *height* of δ is the integer l given by $l = \text{rank}(\mathfrak{U}) - \text{rank}(\mathfrak{L})$. The matroids are the delta matroids of height zero.

REMARK. In a weaker meaning, Proposition 1.1 states that if δ is even, then (BP) is satisfied if and only if δ is equivalent to a matroid. The exact property corresponding to Proposition 1.1 is:

(BP″) There exists $F \in \mathscr{F}$ such that G_F^δ is bipartite.

(BP″) is closed by equivalence, but the minimal height of an equivalent of δ is not bounded, as can be seen by the following example. Let $\delta = (V, \mathscr{F})$ be the delta matroid defined by $\mathscr{F} = \mathscr{P}(V) - \{X \subseteq V : |X| = 2\}$. Then δ satisfies (BP″) with $F = \varnothing$. If $|V| = n \geqslant 5$, then for $|X| \neq 2$ and $|X| \neq n-2$, the height of $\delta \vartriangle X$ is n, and for $|X| = 2$ or $|X| = n-2$, the height is $n-1$.

Our main results are the following,

PROPOSITION 1.3. *A delta matroid δ satisfies* (BP′) *if and only if one of the following statements holds*:

(i) *The height l of δ satisfies $l \leqslant 1$.*
(ii) *δ does not contain as minors*

$$\delta_1 = (\{a,b\},\{\varnothing, ab\}), \qquad \delta_2 = (\{a,b\},\{\varnothing, a, ab\}), \quad \textit{and}$$

$$\delta_3 = (\{a,b\},\{\varnothing, a, b, ab\}).$$

PROPOSITION 1.4. *A delta matroid δ satisfies* (BP) *if and only if it is equivalent to some delta matroid δ' such that one of the following statements holds*:

(i) *The height l of δ' satisfies $l \leqslant 1$.*
(ii) Env(δ') *is the direct sum of two delta matroid whose height is* 1.

PROPOSITION 1.5. *δ is an obstruction to* (BP) *if and only if δ is equivalent to one of the delta matroids on $\{x, y, z\}$ having on the following for base sets*:

$$\{\varnothing, xy, yz, zx\}, \qquad \{\varnothing, xy, yz, zx, xyz\}, \qquad \{\varnothing, x, xy, yz, zx, xyz\},$$

$$\{\varnothing, x, y, xy, yz, zx, xyz\}, \qquad \{\varnothing, x, y, z, xy, yz, zx, xyz\}.$$

2. DEFINITIONS AND PRELIMINARIES

(a)

To any delta matroid $\delta = (V, \mathscr{F})$, we associate a *symmetric matroid* [1] $\mathfrak{S} = (W, \mathscr{T})$ on the *symmetric set* $W = V \cup V^{\sim}$, whose *base set* $\mathscr{T}$ is $\{F \cup (V - F)^{\sim} : F \in \mathscr{F}\}$ (the function $x \mapsto x^{\sim}$ is an *involution without a fixed point*).

Let $\mathsf{T}(W) := \{X \subseteq W : |X \cap \{x, x^{\sim}\}| \leqslant 1\}$ be the set of *subtransversals* of W. We have $\mathscr{T} \subseteq \mathsf{T}(W)$. $X, Y \subseteq W$ are *compatible* if $X \cup Y \in \mathsf{T}(W)$. $X \subseteq W$ is an *independent set* if there exists a base containing X. $X \in \mathsf{T}(W)$ is a *circuit* if X is a minimal nonindependent set. X is a *cocircuit* if $X^{\sim}$ is a circuit. Note that a symmetric matroid can also be defined by its independent sets or circuits as follows:

Definition 2.1. $\mathscr{I} \subseteq \mathsf{T}(W)$ is the *collection of independent sets* of a symmetric matroid on W if and only if:

(I1) $\varnothing \in \mathscr{I}$.
(I2) If $I \in \mathscr{I}$ and $x \in W$, then $I + x \in \mathscr{I}$ or $I + x^{\sim} \in \mathscr{I}$.
(I3) If $J \in \mathscr{I}$ and $I \subseteq J$, then $I \in \mathscr{I}$.
(I4) If $I_1, I_2 \in \mathscr{I}$, I_1, I_2 compatible and $|I_1| < |I_2|$, then there exists $x \in I_2 - I_1$ such that $I_1 + x \in \mathscr{I}$.

Definition 2.2. $\mathscr{C} \subseteq \mathsf{T}(W)$ is the *circuit set* of a symmetric matroid on W if and only if

(C1) $\varnothing \notin \mathscr{C}$.
(C2) If $C_1, C_2 \in \mathscr{C}$, then $|C_1 \cap C_2^{\sim}| \neq 1$.
(C3) If $C_1, C_2 \in \mathscr{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
(C4) If C_1 and C_2 are distinct and compatible members of $\mathscr{C}$ and $x \in C_1 \cap C_2$, then there exists $C \in \mathscr{C}$ such that $C \subseteq C_1 \cup C_2 - x$.

The equivalence of these axioms with axiom (SEA) is easy to prove.

(I4) and (C4) are respectively the *compatible augmentation axiom* and the *compatible elimination axiom* of [1].

These notions of circuit, cocircuit, and independent set have an immediate interpretation in the delta matroid δ. Let $D(V)$ be the set $\{(X, Y) : X, Y \subseteq V,\ X \cap Y = \varnothing\}$. (X, Y) is said to be *included in* (X', Y') if $X \subseteq X'$ and $Y \subseteq Y'$. $(X, Y) \in D(V)$ is an *independent set* of δ if there exists $F \in \mathscr{F}$ such that $X \subseteq F$ and $F \cap Y = \varnothing$, else (X, Y) is a *dependent set*. A *circuit* is a minimal dependent set, and (X, Y) is a *cocircuit* if (Y, X) is a circuit. Thus $(X, Y) \in$

$D(V)$ is a circuit (cocircuit, independent set) of δ if and only if $X \cup Y^{\sim}$ is a circuit (cocircuit, independent set) of $\mathfrak{S}$.

(b)

By interpretation in $\mathfrak{S}$ and by (C2), we obtain

PROPOSITION 2.3. *X is a circuit of the upper matroid $\mathfrak{U}$ (respectively the lower matroid $\mathfrak{L}$) if and only if $(X,\emptyset)$ (respectively $(\emptyset, X)$) is a circuit of δ.*

PROPOSITION 2.4. *We have $\mathfrak{U} \to \mathfrak{L}$, i.e., $\mathfrak{L}$ is a quotient of $\mathfrak{U}$.*

We define the *envelope* of δ to be the delta matroid on V, denoted by $\mathrm{Env}(\delta)$, whose base set is $\{X \subseteq V: \exists H \in \mathscr{H}, \exists B \in \mathscr{B}$ such that $B \subseteq X \subseteq H\}$. Such a delta matroid is said to be *full*. These delta matroids are the *g-matroids* of E. Tardos.

In connection with Proposition 2.3, we have the following characterization of a full delta matroid.

PROPOSITION 2.5. *δ is full if and only if for any circuit (P,Q), $P=\emptyset$ or $Q=\emptyset$.*

Proof. Suppose that δ is full. Then (X,Y) is a dependent set if and only if X is a dependent set of $\mathfrak{U}$ or Y is a dependent set of the dual matroid $\mathfrak{L}^*$. So (X,Y) contains $(C,\emptyset)$ with C a circuit of $\mathfrak{U}$ or contains $(\emptyset, D)$ with D a cocircuit of $\mathfrak{L}$. Therefore, if (P,Q) is a circuit of δ, then we have $P=\emptyset$ or $Q=\emptyset$.

Conversely, suppose that any circuit satisfies this property. If $F', F'' \in \mathscr{F}$ and if $F \subseteq V$ are such that $F' \subseteq F \subseteq F''$, then we have $F \in \mathscr{F}$. Else, we may assume that $F = F' + x \notin \mathscr{F}$ with $x \in F'' - F'$. Also, there exists a circuit (P,Q) such that $P \subseteq F' + x$ and $Q \cap (F'+x) = \emptyset$. Necessarily $x \in P$ and hence $Q=\emptyset$. But then we have $(P,Q) \subseteq (F'', V - F'')$ which is a contradiction. Hence δ is full. ■

(c)

A *fundamental circuit with respect to $F \in \mathscr{F}$* is a circuit (P,Q) of δ such that $|P-F|=1$ and $Q \cap F = \emptyset$. It is easy to prove

PROPOSITION 2.6. *If (P,Q) is a fundamental circuit with respect to $F \in \mathscr{F}$ and if $P - F = x$, then $P = \{y \in V: F - x + y \in \mathscr{F}\}$ and $Q = \{y \in V: F + x + y \in \mathscr{F}\}$.*

(d)

From the notion of minors of a symmetric matroid [7] we deduce the notion of minors of a delta matroid (see also [4]). We only recall their definitions. For $A \subseteq E$, we have

(1) the *contraction* $\delta/A = (V - A, \mathscr{F}/A)$ where $\mathscr{F}/A = \{F - A : F \in \mathscr{F}$ and $F \cap A$ is maximal$\}$.

(2) the *restriction* $\delta \backslash A = (V - A, \mathscr{F} \backslash A)$ where $\mathscr{F} \backslash A = \{F - A : F \in \mathscr{F}$ and $F \cap A$ is minimal$\}$.

(3) the *projection* $\delta(A) = (A, \mathscr{F}(A))$ where $\mathscr{F}(A) = \{F \cap A : F \in \mathscr{F}\}$.

The properties of these minors are similar to properties of minors of matroids and are used without proof. They can be deduced from corresponding properties of symmetric matroids [7]. For example, here is a characterization of a full delta matroid using minors.

Proposition 2.7. *δ is full if and only if δ does not contain as minors*

$$\delta_1 = (\{a,b\}, \{\varnothing, ab\}) \quad \textit{and} \quad \delta_2 = (\{a,b\}, \{\varnothing, a, ab\}).$$

Proof. Assume that δ contains δ_1 or δ_2. Then there exist $F', F'' \in \mathscr{F}$ such that $\delta/F' \backslash (V - F'')$ is isomorphic to δ_1 or δ_2. Hence, we have $F'' = F' + \{a, b\}$ and $F' + b \notin \mathscr{F}$, and so δ is not full. Conversely, if δ is not full, then there exist $F', F'' \in \mathscr{F}$ and $X \subseteq V$ such that $F' \subseteq X \subsetneq F''$ and $X \notin \mathscr{F}$. Choose those F', F'', and X with $|X - F'| + |F'' - F'|$ minimum. Necessarily, we have $X = F' + x$ for $x \in F'' - F'$, and by (SEA), $|F'' - F'| = 2$. Thus $\delta/F' \backslash (V - F'')$ is isomorphic to δ_1 or δ_2. ∎

3. CHARACTERIZATIONS FOR (BP′) AND (BP)

First, here is a direct proof of Proposition 1.1 of A. Bouchet:

Proof of Proposition 1.1. By Proposition 1.2, the condition is sufficient. We show that it is necessary. Suppose that $G = G_F^{\delta}$ is bipartite, and let F' be a color class. For $\delta' = \delta \vartriangle F \vartriangle F'$ we have $G = G_{F \vartriangle F \vartriangle F'}^{\delta \vartriangle F \vartriangle F'} = G_{F'}^{\delta'}$ with $F' \in \mathscr{F} \vartriangle F \vartriangle F'$ being both a base of δ' and a color class of G. Now, if $H \in \mathscr{H}$ and $B \in \mathscr{B}$ satisfy $B \subseteq F' \subseteq H$, we necessarily have $B = F' = H$. Indeed, for example, suppose that $|H - F'| = n \geqslant 1$. Then $n = 1$ or 2 is impossible, because δ' is even and F' a color class of G. Hence, we have $n \geqslant 3$.

Then by (SEA), there exists a base F'' of δ' with $F' \subseteq F'' \subseteq H$ and $|F'' - F'| = 2$, but this contradicts the fact that G is bipartite. By duality, $B = F'$. ■

PROPOSITION 3.1. *The family of delta matroids satisfying* (BP) *(or* (BP′)*) is closed by minors.*

Proof. Let $\delta = (V, \mathscr{F})$ be a delta matroid satisfying (BP) [or (BP′)] and $A \subseteq V$.

(1) *Minor* δ / A. Denote $\delta' = \delta / A$ and $\mathscr{F}' = \mathscr{F} / A$. Let B be a maximal intersection of A with a base F of δ. Then we verify that $F' \in \mathscr{F}'$ if and only if $F' \subseteq V - A$ and $F' \triangle B \in \mathscr{F}$. Thus, for $x, y \in V - A$ and $x \neq y$, $F' \triangle \{x, y\} \in \mathscr{F}'$ is equivalent to $F' \triangle B \triangle \{x, y\} \in \mathscr{F}$, and so $G_{F'}^{\delta'}$ is the subgraph of $G_{F' \cup B}^{\delta}$ induced by $V - A$.

(2) *Minor* $\delta \backslash A$. This results from (1) by duality.

(3) *Minor* $\delta(A)$. Denote $\delta' = \delta(A)$ and $\mathscr{F}' = \mathscr{F}(A)$. Let $F' \in \mathscr{F}'$ and $F \in \mathscr{F}$ such that $F' = F \cap A$. Then $G_{F'}^{\delta'}$ is the subgraph of G_F^{δ} induced by A. Hence the result. ■

For example, $\delta_0 = (\{x, y, z\}, \{\varnothing, x, y, z, xyz\})$ is a forbidden minor for (BP).

Proof of Proposition 1.3. (BP′) $\Rightarrow$ (i): Suppose that δ satisfies (BP′). By Proposition 2.6, for any base F and any fundamental circuit (P, Q) associated to F, we have $Q = \varnothing$. Hence, for any circuit (P, Q) of δ we have $P = \varnothing$ or $Q = \varnothing$, and so, by Proposition 2.5, δ is full. Now, let $B \in \mathscr{B}$ and $H \in \mathscr{H}$ such that $B \subseteq H$. By (BP′) and since δ is full, necessarily we have $|H - B| \leqslant 1$. Hence (i).

(i) $\Rightarrow$ (ii) by Definition 2.2, since δ_i does not satisfy (BP′) for $i = 1, 2, 3$ (choose $F = \varnothing$).

(ii) $\Rightarrow$ (i). Suppose (ii). Then by Proposition 2.7, δ is full. Furthermore, let $B \in \mathscr{B}$ and $H \in \mathscr{H}$ such that $B \subseteq H$. If $|H - B| \geqslant 2$, for distinct $a, b \in H - B$, $B + a$ and $B \cup \{a, b\}$ are bases of δ; thus we have $\delta / B \backslash (V - (B \cup \{a, b\}) = \delta_3$, a contradiction. Hence (i).

(i) $\Rightarrow$ (BP′) is easy. ■

The proof of Proposition 1.4 needs six preparatory lemmas. The first one shows that condition (ii) is sufficient.

LEMMA 1. *If for a delta matroid* δ, $\mathrm{Env}(\delta)$ *is the direct sum of delta matroids* $\delta_i = (V_i, \mathscr{F}_i)$, *each one having height* 1 $(i = 1, 2)$, *then* δ *satisfies* (BP).

Proof. For $i = 1, 2$, denote $\mathscr{H}_i = \max \mathscr{F}_i$ and $\mathscr{B}_i = \min \mathscr{F}_i$. Then the height of δ is 2, and we have $V = V_1 \cup V_2$, $\mathscr{H} = \{H_1 \cup H_2 : H_i \in \mathscr{H}_i\}$, $\mathscr{B} = \{B_1 \cup B_2 : B_i \in \mathscr{B}_i\}$. Let $\mathscr{I} = \mathscr{F} - (\mathscr{H} \cup \mathscr{B})$ be the *middle floor*. For $F \in \mathscr{F}$, we show that $G = G_F^\delta$ is bipartite.

(1) If $F \in \mathscr{I}$, then $xy \in G$ implies $|F \cap \{x, y\}| = 1$. Thus G is bipartite and F is a color class.

(2) If $F \in \mathscr{H}$, then $F = F_1 \cup F_2$ with $F_i \in \mathscr{H}_i$. For $xy \in G$, we have $F' = F \vartriangle \{x, y\} \in \mathscr{F}$ and exactly one of the two possible cases:

(a) If $F' \in \mathscr{H}$, then $x, y \in V_1$ or $x, y \in V_2$. Else, for example, we have $x \in V_1$ and $y \in V_2$, hence $F' = (F_1 \vartriangle x) \cup (F_2 \vartriangle x)$ with necessarily $F_1 \vartriangle x \in \mathscr{B}_1$ and $F_2 \vartriangle y \in \mathscr{B}_2$; but this contradicts $F' \in \mathscr{H}$. Moreover $x, y \in V_i$ implies $|\{x, y\} \cap F_i| = 1$, for $i = 1, 2$.

(b) If $F' \in \mathscr{B}$, we similarly show that if $xy \in G$ then $|\{x, y\} \cap F| = 1$ for $i = 1, 2$.

Now, G is union of three bipartite graphs, respectively on $(F_i, V_i, - F_i)$ for $i = 1, 2$, and on (F_1, F_2), and since the bipartitions (V_1, V_2), $(V_1 \cup F_2, V_2 - F_2)$, and $(V_2 \cup F_1, V_1 - F_1)$ define cocycles of G, all cycles of G have an even cardinal. Hence G is bipartite.

(3) If $F \in \mathscr{B}$, by duality we have the same result. ∎

The second lemma generalizes the argument used in the direct demonstration of Proposition 1.1.

LEMMA 2. *Let δ be a delta matroid satisfying* (BP), *and $F' \in \mathscr{F}$ be a base such that $G_{F'}^\delta$ is bipartite, F' being a color class. Then the height l of δ satisfies $l \leqslant 2$.*

Proof. Denote $G = G_{F'}^\delta$, and consider $H \in \mathscr{H}$, $B \in \mathscr{B}$ such that $B \subseteq F' \subseteq H$. We have $|H - F'| \leqslant 1$ and $|F' - B| \leqslant 1$. Else, we have for example, $|H - F'| > 1$. Let $x \in H - F'$. If $F_1 = F' + x \notin \mathscr{F}$, by (SEA), there exists $y \in H - F'$ such that $y \neq x$ and $F' \vartriangle \{x, y\} \in \mathscr{F}$. Then $xy \in G$, which contradicts the fact that G is bipartite with $V - F'$ as a color class. Thus, we have $F_1 \in \mathscr{F}$. We choose F'' minimal for the property $F_1 \subseteq F'' \subseteq H$, $F'' \in \mathscr{F}$, and $F'' \neq F_1$. (SEA) implies $|F'' - F_1| \leqslant 2$ and, since G is bipartite on $(F', V - F')$, more exactly, $|F'' - F_1| = 2$. Now, we obtain $F'' = F' \cup \{x, y, z\}$ with pairwise distinct x, y, z, and the minor $\delta / F' \backslash (V - F'')$ is isomorphic to δ_0 that does not satisfy (BP), in contradiction with Proposition 3.1. By duality, we have $|F' - B| \leqslant 1$ and so $l \leqslant 2$. ∎

Now, we assume that δ satisfies (BP) and $l = 2$.

We consider the quotient of matroids $\mathfrak{U} \to \mathfrak{L}$ associated to Env(δ). A *line* D of $\mathfrak{L}$ is a union $C_1 \cup C_2$, where $\{C_1, C_2\}$ is a modular pair of distinct circuits of $\mathfrak{L}$ (see [11]). A line D or a circuit C of $\mathfrak{L}$ that is an independent set of $\mathfrak{U}$ is said to be *distinguished*.

A nondistinguished circuit of $\mathfrak{L}$ is necessarily a circuit of $\mathfrak{U}$. The family $\mathscr{E}$ of these circuits is a *linear subclass* of $\mathfrak{L}$, i.e., if $\{C_1, C_2\}$ is a modular pair with $C_i \in \mathscr{E}$, then any circuit $C \subseteq C_1 \cup C_2$ satisfies $C \in \mathscr{E}$. Indeed,

PROPOSITION 3.2. *If $\mathfrak{U} \to \mathfrak{L}$ is a quotient of matroids on V, the family $\mathscr{E}$ of mutual circuits of $\mathfrak{U}$ and $\mathfrak{L}$ is a linear subclass of $\mathfrak{L}$.*

Proof. A modular pair $\{C_1, C_2\}$ of $\mathscr{E}$ is modular in $\mathfrak{U}$, because if B is a base of $\mathfrak{L}$ such that $|C_i - B| = 1$ and if H is a base of $\mathfrak{U}$ satisfying $B \subseteq H$, then we necessarily have $|C_i - H| = 1$ $(i = 1, 2)$.

Let C be a circuit of $\mathfrak{L}$, $C \subseteq C_1 \cup C_2$. We have $(C_1 \cup C_2) - C \neq \varnothing$; hence for some $x \in (C_1 \cup C_2) - C$, since $\{C_1, C_2\}$ is modular in $\mathfrak{L}$, C is the only circuit of $\mathfrak{L}$ such that $C \subseteq (C_1 \cup C_2) - x$. Let S be the only circuit of $\mathfrak{U}$ satisfying $S \subseteq (C_1 \cup C_2) - x$. Since $\mathfrak{U} \to \mathfrak{L}$, S is the union of circuits of $\mathfrak{L}$ and so $C = S$. Hence $C \in \mathscr{E}$. ∎

Since the height l of δ is equal to 2, a distinguished line can be seen as a pair $\{B, H\}$, $B \in \mathscr{B}$ and $H \in \mathscr{H}$, such that $B \subseteq H$. Then if $H - B = \{a, b\}$, C_1, C_2 are the fundamental circuits of $\mathfrak{L}$ respectively defined by a, b with respect to the base B.

LEMMA 3. *Let $D = C_1 \cup C_2$ be a distinguished line. Then C_1, C_2 are not connected in $\mathfrak{L}$.*

Proof. Consider $a \in C_1 - C_2$, $b \in C_2 \cup C_1$, H, B defined as above, and $(C(x), x \in V - B)$ the family of fundamental circuits of $\mathfrak{L}$ with respect to B. If a, b are connected, there exists a sequence $(x_1, \ldots, x_n)$, $x_i \in V - B$, such that $x_1 = a$, $x_n = b$, and $C(x_i) \cap C(x_{i+1}) \neq \varnothing$ for $i = 1, \ldots, n-1$. Then we choose $y_i \in C(x_i) \cap C(x_{i+1})$. For $i = 1, \ldots, n-1$, we have $B + x_i - y_i \in \mathscr{B}$ and $B + x_{i+1} - y_i \in \mathscr{B}$. Hence, $x_i y_i, x_{i+1} y_i \in G = G_B^\delta$ and $(a = x_1, y_1, \ldots, x_i, y_i, \ldots, y_{n-1}, x_n = b)$ is an even chain of G. Since $H = B \cup \{a, b\} \in \mathscr{H}$, we have $ab \in G$; hence G contains an odd cycle, so we have a contradiction. ∎

Thus C_1, C_2 belong to distinct connected components of $\mathfrak{U}$. We say that these components *cut* the line. Let Γ be the simple graph whose vertex set is the collection of connected components of $\mathfrak{L}$, two components K, K' giving an edge KK' if and only if K, K' cut a distinguished line.

LEMMA 4. *Let* $D = C_1 \cup C_2$ *be a distinguished line, and* C_3 *be a distinguished circuit connected to* C_2. *Then* $C_1 \cup C_3$ *is a distinguished line.*

Proof. By Proposition 3.2, the set of nondistinguished circuits is a linear subclass $\mathscr{E}$ of $\mathfrak{L}$. Since C_2, C_3 are connected and $C_2, C_3 \notin \mathscr{E}$, by a proposition of W. T. Tutte [11, 4.34], C_2 can be reached along a *chain* from C_3, a chain made up of circuits of $\mathfrak{L}$ which do not belong to $\mathscr{E}$. So we can assume that $D' = C_2 \cup C_3$ is a line. By Lemma 3, D' is not distinguished; hence, since C_2, C_3 are independent sets, D' is a circuit of $\mathfrak{U}$. Since D' is a line, we can suppose that $|C_3 - B| = 1$. Then, D' is a fundamental circuit with respect to the base $H = B \cup (C_1 \cup C_2)$ of $\mathfrak{U}$; hence $D'' = C_1 \cup C_3$ is a distinguished line. ∎

LEMMA 5. Γ *is bipartite.*

Proof. Else, let $(F_1, \ldots, F_{2n+1})$ be a minimal odd cycle of Γ. By Lemma 4 there exist circuits $C_i \subseteq F_i$ such that $C_i \cup C_{i+1}$ is a distinguished line (for $i = 1, \ldots, 2n$), and $C_{2n+1} \cup C_1$ likewise. Choose $a_i \in C_i$ $(i = 1, \ldots, 2n+1)$. Since $C_i - a_i$ are independent sets of $\mathfrak{L}$ and since F_i are connected components of $\mathfrak{L}$, there exists points a_i and a base B of $\mathfrak{L}$ such that for all i, $C_i - B = a_i$ holds. We show that $H_i = B \cup \{a_i, a_{i+1}\} \in \mathscr{H}$ for $i = 1, \ldots, 2n$. Else H_i contains a circuit S_i of $\mathfrak{U}$ which is, since $\mathfrak{U} \to \mathfrak{L}$, a union of circuits of $\mathfrak{L}$. Thus, for the distinguished line $D = C_i \cup C_{i+1}$, we have $S_i \subseteq D$, which is a contradiction. Similarly, we have $H_{2n+1} = B \cup \{a_{2n+1}, a_1\} \in \mathscr{H}$. So $\{a_{2n+1}, a_1\}$ and $\{a_i, a_{i+1}\}$ are edges of G_B^δ $(i = 1, \ldots, 2n)$, and this gives an odd cycle, whereas G_B^δ is bipartite. ∎

Let (V_1, V_2) be the partition of V corresponding to a partition of the vertex set of Γ.

LEMMA 6. *Let* $C_i \subseteq V_i$ *for* $i = 1, 2$ *be distinguished circuits. Then* $D = C_1 \cup C_2$ *is a distinguished line.*

Proof. Consider $a_i \in C_i$ and a base B_i of V_i, in $\mathfrak{L}$, such that $C_i - B_i = a_i$ $(i = 1, 2)$. We have $B = B_1 \cup B_2 \in \mathscr{B}$, and, since any circuit of $\mathfrak{U}$ is a union of circuits of $\mathfrak{L}$ and C_i is independent in $\mathfrak{U}$, $B \cup C_i$ is independent. Also, let $H \in \mathscr{H}$ be a base such that $B \cup C_1 \subseteq H$. We can assume that $a_2 \in H$. Indeed, if not, then since $|H - B| = 2$, there exists $a_3 \in H - (B \cup C_1)$ with $a_3 \neq a_2$. Let C_3 be the fundamental circuit of a_3 with respect to B. Since $C_1 \cup C_3$ is distinguished, we necessarily have $C_3 \subseteq V_2$. Furthermore, since $B \cup C_2$ is independent in $\mathfrak{U}$, there exists $x \in H - (B \cup C_2)$ with $H - x +$

$a_2 \in \mathscr{H}$. But we have $H-(B\cup C_2)=\{a_1,a_3\}$, and $x=a_1$ is impossible, else $D'=C_2\cup C_3$ is a distinguished line with $C_2,C_3\in V_2$, which contradicts the definitions of Γ and (V_1,V_2). So $x=a_3$, and $H'=H-x+a_2$ is suitable.

Thus $B\cup C_1\cup C_2\subseteq H$, and by definition $D=C_1\cup C_2$ is a distinguished line. ∎

Proof of Proposition 1.4. By Propositions 1.2, 1.3 and Lemma 1, each of conditions (i) and (ii) is sufficient.

Conversely, assume that $\delta=(V,\mathscr{F})$ satisfies (BP). Consider $F\in\mathscr{F}$ and the bipartite graph G_F^δ, with the color class F'. We have $G_F^\delta=G_{F\Delta F\Delta F'}^{\delta\Delta F\Delta F'}$ and $F'\in\mathscr{F}\Delta F\Delta F'=\mathscr{F}'$. Hence, if $\delta'=\delta\Delta F\Delta F'$, then F' is a color class of the bipartite graph $G_{F'}^{\delta'}$ and δ' satisfies (BP), by Proposition 1.2. So, by Lemma 2, the height l of δ' satisfies $l\leqslant 2$, and hence we can assume that $F'=F$ and $\delta'=\delta$.

If $l\leqslant 1$, (i) holds. We suppose $l=2$, and we prove that $\operatorname{Env}(\delta)$ is a direct sum of two delta matroids whose height is 1.

Lemmas 3 to 6 may be applied. Then, let (V_1,V_2) be the obtained partition of V, and denote by $\mathfrak{U}_i$, $\mathfrak{L}_i$ respectively the restrictions of $\mathfrak{U}$ and $\mathfrak{L}$ to V_i $(i=1,2)$. By construction, we have $\mathfrak{L}=\mathfrak{L}_1\oplus\mathfrak{L}_2$, $\operatorname{rank}(\mathfrak{U}_i)=\operatorname{rank}(\mathfrak{L}_i)+1$, and clearly, $\mathfrak{U}_i\to\mathfrak{L}_i$ $(i=1,2)$. We show that $\mathfrak{U}=\mathfrak{U}_1\oplus\mathfrak{U}_2$.

Let H_i be a base of $\mathfrak{U}_i$. We have $H_i=B_i\cup C_i$ with C_i a distinguished circuit of $\mathfrak{L}$ and B_i a base of $\mathfrak{L}_i$ $(i=1,2)$. Then by Lemma 6, $H=B_1\cup B_2\cup C_1\cup C_2$ is a base of $\mathfrak{U}$. Hence, the result holds.

Now, if δ_i is the full delta matroid associated to the quotient $\mathfrak{U}_i\to\mathfrak{L}_i$ for $i=1,2$, we have $\operatorname{Env}(\delta)=\delta_1\oplus\delta_2$, and this concludes the proof. ∎

4. OBSTRUCTIONS TO (BP)

An *obstruction* to (BP) is a delta matroid that does not satisfy (BP) and every proper minor of which satisfies (BP).

The following result shows the known relations between minors and equivalent delta matroids of $\delta=(V,\mathscr{F})$.

PROPOSITION 4.1. *Let $A,X\subseteq V$, and denote $A_1=X_1=A\cap X$, $A_2=A-X$, $X_2=X-A$. Then we have* $(\delta\Delta X)/A=(\delta/A_2\backslash A_1)\Delta X_2$, $(\delta\Delta X)\backslash A=(\delta\backslash A_1/A_2)\Delta X_2$, $(\delta\Delta X)(A)=\delta(A)\Delta X_1$.

We verify it by taking elementary minors (see [4, (2.1)]).

PROPOSITION 4.2. *Let* (Q) *be a property closed under minors and equivalence, and let* $\mathscr{E}$ *be the set of obstructions to* (Q). *Then* $\mathscr{E}$ *is closed under equivalence.*

Proof. For $\delta=(V,\mathscr{F})\in\mathscr{E}$ and $X\subseteq V$, we show that $\delta'=\delta\vartriangle X\in\mathscr{E}$. First, since (Q) is closed under equivalence, δ' does not satisfy (Q). If δ' is not minimal, there exists $p\in V$ such that $\delta'/p=(\delta\vartriangle X)/p$ or $\delta'\setminus p=(\delta\vartriangle X)\setminus p$ or $\delta'(V-p)=(\delta\vartriangle X)(V-p)$ does not satisfy (Q). By Proposition 4.1, we have a strict minor δ'' of δ and $Y\subseteq V$ such that $\delta''\vartriangle Y$ does not satisfy (Q). So δ'' does not verify (Q); but this contradicts $\delta\in\mathscr{E}$. ∎

Proof of Proposition 1.5. Let $\mathscr{E}$ be the set of obstructions to (BP), and $\delta=(V,\mathscr{F})\in\mathscr{E}$. Since δ does not satisfy (BP), there exists a base $F\in\mathscr{F}$ such that $G=G_F^{\delta}$ is not bipartite. Since $\mathscr{E}$ is closed under equivalence, we can suppose that $F=\varnothing$ (else we consider the delta matroid $\delta\vartriangle F$).

G contains an elementary odd cycle, and since δ is minimal, in fact G is an elementary odd cycle C without a chord. Thus, we have $V=C=\{a_1,\dots,a_{2n+1}\}$ and $\{\varnothing,a_1,a_2,\dots,a_{2n}a_{2n+1},a_{2n+1}a_1\}\subseteq\mathscr{F}$, with $n\geqslant 1$. We prove $n=1$, the result following by checking of all possible delta matroids.

For $p\in V$, V' denotes $V-p$ and δ' denotes $\delta(V')=(V',\mathscr{F}')$. By minimality of δ, δ' satisfies (BP).

(1) For two nonadjacent edges $u=\{a_i,a_{i+1}\}$, $v=\{a_j,a_{j+1}\}$ of C, we have $\{a_i,a_{i+1},a_j,a_{j+1}\}\in\mathscr{F}$. We can assume that $i+1<j$ holds. Since the cycle C is without chord, we have $\{a_i,a_{j+1}\}\notin\mathscr{F}$ and $\{a_{i+1},a_{j+1}\}\notin\mathscr{F}$. Hence, for $F_1=\{a_i,a_{i+1}\}$, $F_2=\{a_j,a_{j+1}\}$, and $x=a_{j+1}$, (SEA) gives $\{a_i,a_{i+1},a_{j+1}\}\in\mathscr{F}$ or $\{a_i,a_{i+1},a_j,a_{j+1}\}\in\mathscr{F}$. Let μ be the *odd chain* of C *that is adjacent to edges* u,v, and let p be the mutual end vertex of μ and u. Then, for $X=\{a_i,a_{i+1},a_{j+1}\}\in\mathscr{F}$, $\mu-u$ and $X-p$ define an odd cycle of $G_{\varnothing}^{\delta'}$, a contradiction. From now on, we take $p=a_1$. Since δ' satisfies (BP), by Proposition 1.4 there exists $X\subseteq V'$ such that the height l of $\delta''=\delta'\vartriangle X$ satisfies $l\leqslant 2$. Let $\mathscr{F}''$ denote the base set of δ''. We have

$$X,\ X\vartriangle a_2,\ X\vartriangle a_{2n+1}\in\mathscr{F}'',$$

$$X\vartriangle\{a_i,a_{i+1}\}\in\mathscr{F}''\quad\text{for}\quad i=2,3,\dots,2n,\tag{I}$$

$$X\vartriangle\{a_i,a_{i+1},a_j,a_{j+1}\}\in\mathscr{F}''\quad\text{for}\quad 2\leqslant i\leqslant j-2\leqslant 2n-1.$$

(2) $l=2$. For if not, since $X \in \mathscr{F}''$, we have $|X \cap \{a_i, a_{i+1}\}| = 1$ for $i = 2,\dots,2n$. Thus, one of the following statements holds:

(i) $a_2,\dots,a_{2i},\dots,a_{2n} \in X$ and $a_3,\dots,a_{2i+1},\dots,a_{2n+1} \notin X$.
(ii) $a_2,\dots,a_{2i},\dots,a_{2n} \notin X$ and $a_3,\dots,a_{2i+1},\dots,a_{2n+1} \in X$.

Hence, we have $X - a_2, X + a_{2n+1} \in \mathscr{F}''$ or $X + a_2, X - a_{2n+1} \in \mathscr{F}''$, but this contradicts $l \leqslant 1$.

(3) Now, by Proposition 1.4, Env(δ'') is the direct sum of two delta matroids whose height is 1. Let (V_1, V_2) be the corresponding partition of V', and $X_i = X \cap V_i$, $i = 1,2$. For $\delta_i = \delta''(V_i)$, we have the decomposition Env(δ'') $= \delta_1 \oplus \delta_2$. Without loss of generality, we can suppose $x_2 \in V_2$. Let $\mathscr{B}, \mathscr{B}_i$ and $\mathscr{H}, \mathscr{H}_i$ be respectively the minimal base sets and the maximal base sets of δ'', δ_i, for $i = 1,2$, and $\mathscr{I} = \mathscr{F}'' - (\mathscr{B} \cup \mathscr{H})$ be the middle floor. According to the position of X in $\mathscr{F}''$, we consider three cases.

Case 1: $X \in \mathscr{H}$. Then we have $X_i \in \mathscr{H}_i$, and since $a_2 \in X_2$, also $X_2 - a_2 \in \mathscr{B}_2$. Thus, by $X - \{a_2, a_3\} \in \mathscr{B}$ and decomposition, it follows that $a_3 \in X_1$. Similarly, from (I) and induction we deduce $a_{2i} \in X_2$ and $a_{2i+1} \in X_1$ for $1 \leqslant i \leqslant n$. Hence $X = V'$. Now, we have $n = 1$; else by (I), $X - \{a_2, a_3, a_4, a_5\} \in \mathscr{F}''$ contradicts $l = 2$.
Case 2: $X \in \mathscr{B}$. Since $(\delta'')^* = \delta'' \Delta V' = \delta' \Delta (V' - X)$, by duality and taking $V' - X$, we come back to case 1.
Case 3: $X \in \mathscr{I}$. By duality (and taking $V' - X$, if necessary), we can suppose that $a_2 \in X$. By decomposition, we have $X_2 - a_2 \in \mathscr{B}_2$, $X_1 \in \mathscr{B}_1$, and $X_2 \in \mathscr{H}_2$. Thus, since $X \Delta \{a_2, a_3\} \in \mathscr{F}''$, we have $a_3 \notin X$; hence $a_3 \in V_1 - X_1$. Similarly, we deduce from (I) and induction that $a_{2i} \in X_2$ and $a_{2i+1} \in V_1 - X_1$ for $1 \leqslant i \leqslant n$. Now $n = 1$; else by (I), we have $X \cup \{a_3, a_5\} - \{a_2, a_4\} \in \mathscr{F}''$, which contradicts $l = 2$. ∎

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